IMPACT: International Journal of Research in Engineering & Technology (IMPACT: IJRET)
ISSN(E): 2321-8843; ISSN(P): 2347-4599
Vol. 2, Issue 8, Aug 2014, 77-90

ON INVERSES AND GENERALIZED INVERSES OF BIMATRICES

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ABSTRACT

The concept of singular, Semi-singular and non-singular bimatrices are introduced. The concept of inverse bimatrices, reverse order law and some properties of inverses bimatrices are studied. Also the notion of generalized inverses and some properties of generalized inverse of bimatrices are discussed. The solution of homogeneous and non-homogeneous system of equations are studied.

KEYWORDS: Bimatrix, Inverse Bimatrix, Generalized Inverse of Bimatrix, Singular, Semi- Singular, Non-Singular Bimatrices

AMS Classification: 15A09, 15A15, 15A57.

INTRODUCTION

Let C_{nXn} be the space of nxn complex matrices of order n. For $A\epsilon C_{nXn,}$ let $A^{-1}, A^T, A^*, A^\dagger, r(A), \mathcal{R}(A)$denote the inverse, transpose, conjugate transpose, Moore-penrose inverse, rank and range space of A respectively. A matrix has its inverse if $|A| \neq 0$ that is, A is non singular. Generalized inverse is a great tool in solving linearly dependent and unbalanced system of linear equations. It has the ability to find the solution of square matrix when it is singular and non-square. A solution X of the equation AXA =A is denoted by A^- and is called generalized inverse of A. For $A\epsilon C_{nXn}$, the Moore-penrose inverse $A^\dagger$ of A is the unique solution of the four equation $(i)\ AXA = A, (ii)\ XAX = A, (iii) (AX)^* = AX,\ \ (iv)\ (XA)^* = XA$. The concept of a generalized inverse was first introduced by Fredholm (1903), he called a particular generalized inverse as pseudo inverse which serve as integral operator. However, the concept of an inverse of a singular matrix seems to have been first introduced by Moore [4,5] in 1920. If A_1 and A_2 are any two matrices then the matrix $A_B = A_1 \cup A_2$ is said to be bimatrix [7]. A bimatrix A_B is said to be EP if $N(A_B) = N(A_B^*)$[8].

In this paper the concept of singular, semi-singular, non-singular bimatrices are introduced. The concept of inverse bimatrices, reverse order law and some properties of inverse bimatrices are studied. Also, the notion of generalized inverses and some properties of generalized inverses of bimatrices are discussed. The solution of homogeneous and non-homogeneous system of equations are analysed.

INVERSES OF BIMATRICES

Definition 2.1

Let A_B be a square bimatrix of order n. Then, A_B is said to be invertible if there exists a square bimarix B_B of order n such that

$A_B B_B = B_B A_B = I_B$,

and B_B is called the inverse of A_B and is denoted by A_B^{-1}.

Example 2.2

Let $A_B = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix} \cup \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{pmatrix}$

Now, $A_B^{-1} = \begin{pmatrix} 2 & -1 & -1 \\ -9 & 7 & 4 \\ 5 & -4 & -2 \end{pmatrix} \cup \begin{pmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{pmatrix}$

It is verified that,

$$A_B A_B^{-1} = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix}\begin{pmatrix} 2 & -1 & -1 \\ -9 & 7 & 4 \\ 5 & -4 & -2 \end{pmatrix} \cup \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{pmatrix}\begin{pmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$= I_B$.

Definition 2.3

A square bimatrix A_B is said to be singular if the determinant value of both the components are zero. (That is, $|A_1| = 0$ and $|A_2| = 0$).

Definition 2.4

A square bimatrix $A_B = A_1 \cup A_2$ is said to be non-singular if the determinant value of both the components are non zero.

Definition 2.5

A square bimatrix $A_B = A_1 \cup A_2$ is said to be semi-singular if the determinant value of either one of the component is zero.

Properties of the Conjugate Transpose of Bimatrices 2.6

- $A_B^{**} = A_B$
- $(A_B + B_B)^* = A_B^* + B_B^*$
- $(\lambda A_B)^* = \bar{\lambda} A_B^*$
- $(A_B B_B)^* = B_B^* A_B^*$
- $A_B A_B^* = 0$ implies $A_B = 0$
- $B_B A_B A_B^* = C_B A_B A_B^*$ implies $B_B A_B = C_B A_B$
- $B_B A_B^* A_B = C_B A_B^* A_B$ implies $B_B A_B^* = C_B A_B^*$

Properties of Inverse of Bimatrices 2.7

Let A_B and B_B be the two bimatrices, then the following holds:

- $(A_B B_B)^{-1} = B_B^{-1} A_B^{-1}$
- $(A_B^{-1})^{-1} = A_B$
- $(kA_B)^{-1} = k^{-1} A_B^{-1}$
- $(A_B^t)^{-1} = (A_B^{-1})^t$

Proof of (i)

Given $A_B B_B = (A_1 \cup A_2)(B_1 \cup B_2)$

$= A_1 B_1 \cup A_2 B_2$

$(A_B B_B)^{-1} = (A_1 B_1 \cup A_2 B_2)^{-1}$

$= (A_1 B_1)^{-1} \cup (A_2 B_2)^{-1}$

$= B_1^{-1} A_1^{-1} \cup B_2^{-1} A_2^{-1}$ (since$(AB)^{-1} = B^{-1} A^{-1}$)

$= (B_1^{-1} \cup B_2^{-1})(A_1^{-1} \cup A_2^{-1})$

$(A_B B_B)^{-1} = B_B^{-1} A_B^{-1}$

Proof of (ii)

$A_B^{-1} = (A_1 \cup A_2)^{-1}$

$= A_1^{-1} \cup A_2^{-1}$

$(A_B^{-1})^{-1} = (A_1^{-1} \cup A_2^{-1})^{-1}$

$= (A_1^{-1})^{-1} \cup (A_2^{-1})^{-1}$

$= A_1 \cup A_2$ (since$(A^{-1})^{-1} = A$)

$(A_B^{-1})^{-1} = A_B$

Proof of (iii)

$(kA_B)^{-1} = [k(A_1 \cup A_2)]^{-1}$

$=(kA_1 \cup kA_2)^{-1}$

$=(kA_1)^{-1} \cup (kA_2)^{-1}$

$=k^{-1} A_1^{-1} \cup k^{-1} A_2^{-1}$

$= k^{-1}(A_1^{-1} \cup A_2^{-1})$

$(kA_B)^{-1} = k^{-1} A_B^{-1}$

Proof of (iv)

$A_B^t = (A_1 \cup A_2)^t$

$= A_1^t \cup A_2^t$

$(A_B^t)^{-1} = (A_1^t \cup A_2^t)^{-1}$

$= (A_1^t)^{-1} \cup (A_2^t)^{-1}$

$=(A_1^{-1})^t \cup (A_2^{-1})^t$ (since $(A^t)^{-1} = (A^{-1})^t$)

$=(A_1^{-1} \cup A_2^{-1})^t$

$(A_B^t)^{-1} = (A_B^{-1})^t$

GENERALIZED INVERSES OF BIMATRICES

Definition 3.1

Moore – Penrose inverse of a bimatrix A_B is the unique solution of the following equations:

- $A_B X_B A_B = A_B$
- $X_B A_B X_B = X_B$
- $(A_B X_B)^* = A_B X_B$
- $(X_B A_B)^* = X_B A_B$

Definition 3.2

Group inverse of A_B, denoted as $A_B{}^{\#}$ satisfying the equations,

- $A_B X_B A_B = A_B$.
- $X_B A_B X_B = X_B$.
- $A_B X_B = X_B A_B$.
- If $A_B^{\#}$ exists, then it is unique.

Example 3.3

Let $A_B = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 1 & 5 \\ 3 & 1 & 3 \end{pmatrix} \cup \begin{pmatrix} 1 & 3 & 2 \\ 5 & 2 & 6 \\ 2 & 6 & 4 \end{pmatrix}$

Generalized inverse of A_B is,

$$X_B = \begin{pmatrix} 1/3 & -1/3 & 0 \\ -1/3 & 4/3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cup \begin{pmatrix} -2/13 & 3/13 & 0 \\ 5/13 & -1/13 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

such that $A_B\, X_B\, A_B = A_B$

Lemma 3.4

Let A_B be an $n\times n$ complex bimatrix. Then the following holds :

- $A_B^{\dagger\dagger}=A_B$
- $A_B^{*\dagger}=A_B^{\dagger*}$
- If A_B is non singular, then $A_B^{\dagger}=A_B^{-1}$
- $(\lambda A_B)^{\dagger}=\lambda^{\dagger}A_B^{\dagger}$
- $(A_B^{*}A_B)^{\dagger}=A_B^{\dagger}A_B^{\dagger*}$

Proof of (i)

$A_B^{\dagger}=(A_1\cup A_2)^{\dagger}$

$=A_1^{\dagger}\cup A_2^{\dagger}$

$\left(A_B^{\dagger}\right)^{\dagger}=\left(A_1^{\dagger}\cup A_2^{\dagger}\right)^{\dagger}$

$=\left(A_1^{\dagger}\right)^{\dagger}\cup\left(A_2^{\dagger}\right)^{\dagger}$

$=A_1\cup A_2$ (since $A^{\dagger\dagger}$=A)

$\left(A_B^{\dagger}\right)^{\dagger}=A_B$

Proof of (ii)

$A_B^{*}=(A_1\cup A_2)^{*}$

$=A_1^{*}\cup A_2^{*}$

$(A_B^{*})^{\dagger}=(A_1^{*}\cup A_2^{*})^{\dagger}$

$=(A_1^{*})^{\dagger}\cup(A_2^{*})^{\dagger}$

$=\left(A_1^{\dagger}\right)^{*}\cup\left(A_2^{\dagger}\right)^{*}$ (Since $A^{*\dagger}=(A^{\dagger})^{*}$)

$=\left(A_1^{\dagger}\cup A_2^{\dagger}\right)^{*}$

$=((A_1\cup A_2)^{\dagger})^{*}$

$=\left(A_B^{\dagger}\right)^{*}$

$(A_B^{*})^{\dagger}=\left(A_B^{\dagger}\right)^{*}$

Proof of (iii)

Given A_B is non singular bimatrix $\Rightarrow$ both A_1 and A_2 are non singular matrices.

$A_B^{\dagger} = (A_1 \cup A_2)^{\dagger}$

$= A_1^{\dagger} \cup A_2^{\dagger}$

By lemma (1.3) of [8],

${A_1}^{\dagger} = {A_1}^{-1}$ and $A_2^{\dagger} = {A_2}^{-1}$

$\Rightarrow A_B^{\dagger} = A_1^{-1} \cup A_2^{-1}$

$= (A_1 \cup A_2)^{-1}$

$A_B^{\dagger} = A_B^{-1}$

Proof of (iv)

$(\lambda A_B)^{\dagger} = [\lambda(A_1 \cup A_2)]^{\dagger}$

$= (\lambda A_1 \cup \lambda A_2)^{\dagger}$

$= (\lambda A_1)^{\dagger} \cup (\lambda A_2)^{\dagger}$

$= \lambda^{\dagger} A_1^{\dagger} \cup \lambda^{\dagger} A_2^{\dagger}$ (since $(\lambda A)^{\dagger} = \lambda^{\dagger} A^{\dagger}$)

$= \lambda^{\dagger}\left(A_1^{\dagger} \cup A_2^{\dagger}\right)$

$(\lambda A_B)^{\dagger} = \lambda^{\dagger} A_B^{\dagger}$

Proof of (v)

$A_B^* A_B = (A_1^* \cup A_2^*)(A_1 \cup A_2)$

$= A_1^* A_1 \cup A_2^* A_2$

$(A_B^* A_B)^{\dagger} = (A_1^* A_1 \cup A_2^* A_2)^{\dagger}$

$= (A_1^* A_1)^{\dagger} \cup (A_2^* A_2)^{\dagger}$

$= A_1^{\dagger} A_1^{\dagger *} \cup A_2^{\dagger} A_2^{\dagger *}$ (since $(A^* A)^{\dagger} = A^{\dagger} A^{\dagger *}$)

$= \left(A_1^{\dagger} \cup A_2^{\dagger}\right)\left(A_1^{\dagger *} \cup A_2^{\dagger *}\right)$

$= \left(A_1^{\dagger} \cup A_2^{\dagger}\right)\left(A_1^{\dagger} \cup A_2^{\dagger}\right)^*$

$(A_B^* A_B)^{\dagger} = A_B^{\dagger} A_B^{\dagger *}$

Lemma 3.5 [10]

Let A be an $n \times n$ complex matrix. Then the following statements are equivalent:

- A is EP.
- A^* is EP.

- $r[A, A^*] = r[A]$.
- $A^\dagger$ is EP.
- $AA^\dagger = A^\dagger A$.
- $A^\dagger = A^\#$.

Theorem 3.6

Let A_B be an $n\times n$ complex bimatrix. Then the following are equivalent:

- A_B is EP bimatrix
- $(A_B A_B^\dagger)^2 = A_B^2 (A_B^\dagger)^2$
- $(A_B^\dagger A_B)^2 = (A_B^\dagger)^2 A_B^2$

Proof

Let $A_B = A_1 \cup A_2$ be an $n\times n$ complex bimatrix.

To Prove (i) ⇒(ii)

If A is EP then $AA^\dagger = A^\dagger A$ (By lemma 3.5)

Hence $(A_B A_B^\dagger)^2 = (A_1 A_1^\dagger \cup A_2 A_2^\dagger)^2$

$= (A_1 A_1^\dagger)^2 \cup (A_2 A_2^\dagger)^2$

$= (A_1 A_1^\dagger)\ (A_1 A_1^\dagger) \cup (A_2 A_2^\dagger)(A_2 A_2^\dagger)$

$= A_1(A_1^\dagger\ A_1)A_1^\dagger \cup A_2(A_2^\dagger A_2)A_2^\dagger$

$= A_1(A_1 A_1^\dagger)A_1^\dagger \cup A_2(A_2 A_2^\dagger)A_2^\dagger$

$= A_1^2\ A_1^{\dagger 2} \cup A_2^2\ A_2^{\dagger 2}$

$= (A_1^2 \cup A_2^2)(A_1^{\dagger 2} \cup A_2^{\dagger 2})$

$(A_B A_B^\dagger)^2 = A_B^2\ \ A_B^{\dagger 2}$

To Prove (ii) ⟹(i)

Suppose A_B satisfies (ii)

Note that $(A_B A_B^\dagger)^2 = A_B A_B^\dagger$

That is, $A_B A_B^\dagger = A_B^2 (A_B^\dagger)^2$

$\Rightarrow A_1 A_1^\dagger \cup A_2 A_2^\dagger = A_1^2\ A_1^{\dagger 2} \cup A_2^2\ A_2^{\dagger 2}$

Post-multiply both sides by $A_B A_B^* = A_1 A_1^* \cup A_2\, A_2^*$ we get

$$\left(A_1 A_1^\dagger \cup A_2 A_2^\dagger\right)\left(A_1 A_1^* \cup A_2\, A_2^*\right) = \left(A_1^2\, A_1^{\dagger 2} \cup A_2^2\, A_2^{\dagger 2}\right)\left(A_1 A_1^* \cup A_2\, A_2^*\right)$$

$$A_1 A_1^\dagger A_1 A_1^* \cup A_2 A_2^\dagger\, A_2\, A_2^* = A_1^2\, A_1^{\dagger 2} A_1 A_1^* \cup A_2^2\, A_2^{\dagger 2}\, A_2\, A_2^*$$

Now use the fact that $AA^\dagger A = A$ and $A^\dagger A A^* = A^*$ we get

$$A_1 A_1^* \cup A_2\, A_2^* = A_1^2\, A_1^\dagger\, A_1^* \cup A_2^2\, A_2^\dagger\, A_2^*$$

$$(A_1 \cup A_2)(A_1^* \cup A_2^*) = (A_1^2 \cup A_2^2)\left(A_1^\dagger \cup A_2^\dagger\right)(A_1^* \cup A_2^*)$$

$$A_B\, A_B^* = A_B^2\, A_B^\dagger\, A_B^*$$

Then it follows by equation (1.4) of lemma (1.2) of [10] that

$$r\left(A_B A_B^* - A_B^2 A_B^\dagger A_B^*\right) = r\left[(A_1 A_1^* \cup A_2\, A_2^*) - \left(A_1^2\, A_1^\dagger\, A_1^* \cup A_2^2\, A_2^\dagger\, A_2^*\right)\right]$$

$$= r\left[\left(A_1 A_1^* - A_1^2\, A_1^\dagger\, A_1^*\right) \cup \left(A_2 A_2^* - A_2^2\, A_2^\dagger\, A_2^*\right)\right]$$

$$= r\left(A_1 A_1^* - A_1^2 A_1^\dagger A_1^*\right) \cup r\left(A_2 A_2^* - A_2^2 A_2^\dagger A_2^*\right)$$

$$= r\begin{bmatrix} A_1^* A_1 A_1^* & A_1^* A_1^* \\ A_1^2 A_1^* & A_1 A_1^* \end{bmatrix} - r(A_1) \cup r\begin{bmatrix} A_2^* A_2 A_2^* & A_2^* A_2^* \\ A_2^2 A_2^* & A_2 A_2^* \end{bmatrix} - r(A_2) \quad (3.1)$$

Observe that

$$\begin{bmatrix} A_1^* A_1 A_1^* \\ A_1^2 A_1^* \end{bmatrix}\left(A_1^\dagger\right)^* = \begin{bmatrix} A_1^* A_1 \\ A_1^2 \end{bmatrix};\ \begin{bmatrix} A_2^* A_2 A_2^* \\ A_2^2 A_2^* \end{bmatrix}\left(A_2^\dagger\right)^* = \begin{bmatrix} A_2^* A_2 \\ A_2^2 \end{bmatrix}$$

and $\begin{bmatrix} A_1^* A_1 \\ A_1^2 \end{bmatrix} A_1^* = \begin{bmatrix} A_1^* A_1 A_1^* \\ A_1^2 A_1^* \end{bmatrix};\ \begin{bmatrix} A_2^* A_2 \\ A_2^2 \end{bmatrix} A_2^* = \begin{bmatrix} A_2^* A_2 A_2^* \\ A_2^2 A_2^* \end{bmatrix}$

Hence, $\mathcal{R}\begin{bmatrix} A_1^* A_1 A_1^* \\ A_1^2 A_1^* \end{bmatrix} = \mathcal{R}\begin{bmatrix} A_1^* A_1 \\ A_1^2 \end{bmatrix};\ \mathcal{R}\begin{bmatrix} A_2^* A_2 A_2^* \\ A_2^2 A_2^* \end{bmatrix} = \mathcal{R}\begin{bmatrix} A_2^* A_2 \\ A_2^2 \end{bmatrix}$

According to equation (1.12) of lemma (1.2) in [10], then the equation (3.1) is reduced to

$$\mathrm{r}\left(A_B A_B^* - A_B^2 A_B^\dagger A_B^*\right) = r\begin{bmatrix} A_1^* A_1 & A_1^* A_1^* \\ A_1^2 & A_1 A_1^* \end{bmatrix} - r(A_1) \cup r\begin{bmatrix} A_2^* A_2 & A_2^* A_2^* \\ A_2^2 & A_2 A_2^* \end{bmatrix} - r(A_2)$$

$$= r\left(\begin{bmatrix} A_1^* \\ A_1 \end{bmatrix}[A_1\ A_1^*]\right) - r(A_1) \cup r\left(\begin{bmatrix} A_2^* \\ A_2 \end{bmatrix}[A_2\ A_2^*]\right) - r(A_2)$$

$$= r([A_1\ A_1^*]^*[A_1\ A_1^*]) - r(A_1) \cup r([A_2\ A_2^*]^*[A_2\ A_2^*]) - r(A_2)$$

$$= r([A_1\ A_1^*]) - r(A_1) \cup r([A_2\ A_2^*]) - r(A_2)$$

$$= \{r([A_1\ A_1^*]) \cup r([A_2\ A_2^*])\} - \{r(A_1) \cup r(A_2)\}$$

$$= r\,[A_B\ A_B^*] - r(A_B)$$

$\Rightarrow\ r\left(A_B A_B^* - A_B^2 A_B^\dagger A_B^*\right) = r[A_B\ A_B^*] - r(A_B)$

Hence $A_B A_B^* = A_B^2 A_B^\dagger A_B^*$ is equivalent to $r\ [A_B\ A_B^*] = r(A_B)$

By (i) and (iii) of lemma (3.5),

A_B is EP bimatrix.

To Prove (i) ⇒(iii)

If A is EP then $AA^\dagger = A^\dagger A$ (By lemma 3.5)

Hence $\left(A_B^\dagger A_B\right)^2 = \left(A_1^\dagger A_1 \cup A_2^\dagger A_2\right)^2$

$= \left(A_1^\dagger A_1\right)^2 \cup \left(A_2^\dagger A_2\right)^2$

$= \left(A_1^\dagger A_1\right)\left(A_1^\dagger A_1\right) \cup \left(A_2^\dagger A_2\right)\left(A_2^\dagger A_2\right)$

$= A_1^\dagger(\ A_1 A_1^\dagger)A_1 \cup A_2^\dagger(\ A_2 A_2^\dagger)A_2$

$= A_1^\dagger(A_1^\dagger\ A_1)A_1 \cup A_2^\dagger(A_2^\dagger\ A_2)A_2$

$= A_1^{\dagger 2} A_1^2 \cup\ A_2^{\dagger 2} A_2^2$

$= \left(A_1^{\dagger 2} \cup A_2^{\dagger 2}\right)\left(A_1^2 \cup A_2^2\right)$

$\left(A_B A_B^\dagger\right)^2 = A_B^{\dagger 2}\ A_B^2$

To Prove (iii) ⇒(i)

Suppose A_B satisfies (iii)

We have $\left(A_B A_B^\dagger\right)^2 = A_B A_B^\dagger$

That is, $A_B A_B^\dagger = A_B^2\left(A_B^\dagger\right)^2$

$\Rightarrow A_1 A_1^\dagger \cup A_2 A_2^\dagger = A_1^2\ A_1^{\dagger 2} \cup A_2^2\ A_2^{\dagger 2}$

Pre-multiply both sides by $A_B^*\ A_B = A_1^*\ A_1 \cup\ A_2^*\ A_2$ we get

$(A_1^*\ A_1 \cup\ A_2^*\ A_2)\left(A_1 A_1^\dagger \cup A_2 A_2^\dagger\right) = (A_1^*\ A_1 \cup\ A_2^*\ A_2)\left(A_1^2\ A_1^{\dagger 2} \cup A_2^2\ A_2^{\dagger 2}\right)$

$A_1^*\ A_1(A_1 A_1^\dagger) \cup\ A_2^* A_2\ (A_2 A_2^\dagger) = A_1^* A_1(A_1^2\ A_1^{\dagger 2}) \cup A_2^*\ A_2\ (A_2^2\ A_2^{\dagger 2})$

$A_1^*\ A_1(A_1^\dagger A_1) \cup\ A_2^* A_2\ (A_2^\dagger\ A_2) = A_1^* A_1\left(A_1\ A_1^\dagger\right)^2 \cup A_2^* A_2\left(A_2\ A_2^\dagger\right)^2$

$A_1^*\ A_1 A_1^\dagger A_1 \cup\ A_2^* A_2 A_2^\dagger\ A_2 = A_1^* A_1\left(A_1^\dagger\ A_1\right)^2 \cup A_2^* A_2\left(A_2^\dagger\ A_2\right)^2$

$A_1^*\ A_1 A_1^\dagger A_1 \cup\ A_2^*\ A_2 A_2^\dagger A_2 = A_1^* A_1 A_1^{\dagger 2} A_1^2\ \cup A_2^*\ A_2 A_2^{\dagger 2}\ A_2^2$

Now apply $AA^\dagger A = A$ and $A^* AA^\dagger = A^*$ we get

$\Rightarrow A_1^*A_1 \cup A_2^*A_2 = A_1^*A_1^\dagger A_1^2 \cup A_2^* A_2^\dagger A_2^2$

$(A_1^* \cup A_2^*)(A_1 \cup A_2) = (A_1^* \cup A_2^*)(A_1^\dagger \cup A_2^\dagger)(A_1^2 \cup A_2^2)$

$A_B^* A_B = A_B^* A_B^\dagger A_B^2$

Then it follows by equation (1.4) of lemma (1.2) of [10] that

$r(A_B^* A_B - A_B^* A_B^\dagger A_B^2) = r[(A_1^*A_1 \cup A_2^*A_2) - (A_1^* A_1^\dagger A_1^2 \cup A_2^*A_2^\dagger A_2^2)]$

$= r[(A_1^* A_1 - A_1^*A_1^\dagger A_1^2) \cup (A_2^* A_2 - A_2^*A_2^\dagger A_2^2)]$

$= r(A_1^*A_1 - A_1^*A_1^\dagger A_1^2) \cup r(A_2^*A_2 - A_2^*A_2^\dagger A_2^2)$

$= r\begin{bmatrix} A_1^*A_1A_1^* & A_1^*A_1^2 \\ A_1^*A_1^* & A_1^*A_1 \end{bmatrix} - r(A_1) \cup$

$$r\begin{bmatrix} A_2^*A_2A_2^* & A_2^*A_2^2 \\ A_2^*A_2^* & A_2^*A_2 \end{bmatrix} - r(A_2) \tag{3.2}$$

Observe that

$(A_1^\dagger)^*[A_1^*A_1A_1^* \quad A_1^*A_1^2] = [A_1A_1^* \quad A_1^2]$

and $A_1^*[A_1A_1^* \quad A_1^2] = [A_1^*A_1A_1^* \quad A_1^*A_1^2]$

similarly,

$(A_2^\dagger)^*[A_2^*A_2A_2^* \quad A_2^*A_2^2] = [A_2A_2^* \quad A_2^2]$

and $A_2^*[A_2A_2^* \quad A_2^2] = [A_2^*A_2A_2^* \quad A_2^*A_2^2]$

Hence, $\mathcal{R}[A_1^*A_1A_1^* \quad A_1^*A_1^2] = \mathcal{R}[A_1A_1^* \quad A_1^2]$

and $\mathcal{R}[A_2^*A_2A_2^* \quad A_2^*A_2^2] = \mathcal{R}[A_2A_2^* \quad A_2^2]$

According to equation (1.12) of lemma (1.2) in [10], then the equation(3.2) is reduced to

$\mathrm{r}(A_B^*A_B - A_B^*A_B^\dagger A_B^2) = r\begin{bmatrix} A_1A_1^* & A_1^2 \\ A_1^*A_1^* & A_1^*A_1 \end{bmatrix} - r(A_1) \cup r\begin{bmatrix} A_2A_2^* & A_2^2 \\ A_2^*A_2^* & A_2^* A_2 \end{bmatrix} - r(A_2)$

$= r\left(\begin{bmatrix} A_1 \\ A_1^* \end{bmatrix}[A_1^* \ A_1]\right) - r(A_1) \cup r\left(\begin{bmatrix} A_2 \\ A_2^* \end{bmatrix}[A_2^* \ A_2]\right) - r(A_2)$

$= r([A_1^* \ A_1]^*[A_1^* \ A_1]) - r(A_1) \cup r([A_2^* \ A_2]^*[A_2^* \ A_2]) - r(A_2)$

$= r([A_1^* \ A_1]) - r(A_1) \cup r([A_2^* \ A_2]) - r(A_2)$

$= \{r([A_1^* \ A_1]) \cup r([A_2^* \ A_2])\} - \{r(A_1) \cup r(A_2)\}$

$= r\ [A_B^* \ A_B] - r(A_B)$

$\Rightarrow r(A_B^*A_B - A_B^*A_B^\dagger A_B^2) = r[A_B^* \ A_B] - r(A_B)$

Hence $A_B^* A_B = A_B^* A_B^\dagger A_B^2$ is equivalent to $r\,[A_B^*\, A_B] = r(A_B)$

$\Rightarrow$ A_B is EP bimatrix.

Theorem 3.7

Let $A_B = A_1 \cup A_2$ be a bimatrix, then the four equations

$$A_B\, X_B\, A_B = A_B \quad (3.3)$$

$$X_B\, A_B\, X_B = X_B \quad (3.4)$$

$$(A_B\, X_B)^* = A_B\, X_B \quad (3.5)$$

$$(\, X_B A_B)^* = X_B A_B \quad (3.6)$$

have a unique solution for any A_B.

Proof

First to show that equations (3.4) and (3.5) are equivalent to the single equation,

$$X_B\, X_B^*\, A_B^* = X_B$$

On Substituting (3.5) in (3.4) we get,

$$X_B(A_B\, X_B) = X_B$$

$$X_B(A_B\, X_B)^* = X_B \quad \text{(Since by (3.5))}$$

$$X_B\, X_B^*\, A_B^* = X_B \quad (3.7)$$

Conversely,

$$(3.7) \Rightarrow X_B\, X_B^*\, A_B^* = X_B$$

$$A_B\, X_B\, X_B^*\, A_B^* = A_B\, X_B$$

$$(A_B\, X_B)(A_B\, X_B)^* = A_B\, X_B$$

$$(A_B\, X_B)\,(A_B\, X_B) = A_B\, X_B \quad \text{(Since by (3.5))}$$

$$X_B\, A_B\, X_B = X_B$$

which gives (3.4)

Thus, (3.4) and (3.5) are equivalent to $X_B X_B^* A_B^* = X_B$

Similarly from (3.3) and (3.6)

$$X_B\, A_B\, A_B^* = (\, X_B\, A_B)^* A_B^*$$

$$= A_B^*\, X_B^*\, A_B^*$$

$$= (A_B\, X_B\, A_B)^*$$

$X_B\, A_B\, A_B^* = A_B^*$ (3.8)

Thus, it is sufficient to find an X_B satisfying (3.7) and (3.8). Such an X_B exists if a B_B can be found satisfying,

$B_B A_B^*\, A_B A_B^* = A_B^*$

For then $X_B = B_B A_B^*$ satisfies (3.8). Also we have seen that (3.8) implies,

$A_B^*\, X_B^*\, A_B^* = A_B^*$

and therefore $B_B A_B^*\, X_B^*\, A_B^* = B_B A_B^*$

Thus, X_B also satisfies (3.7).

Now, the expressions $(A_B^* A_B), (A_B^*\, A_B)^2, (A_B^*\, A_B)^3, \ldots$ cannot all be linearly independent that is, there exists a relation

$\lambda_1 A_B^*\, A_B + \lambda_2 (A_B^*\, A_B)^2 +, \ldots, + \lambda_\kappa (A_B^*\, A_B)^\kappa = 0$, (3.9)

where $\lambda_1, \lambda_2, \ldots, \lambda_\kappa$ are not all zero.

Let λ_r be the first non zero λ and put

$B_B = -\lambda_r^{-1}\{\lambda_{r+1} I + \lambda_{r+2}(A_B^*\, A_B) + \cdots + \lambda_\kappa (A_B^*\, A_B)^{k-r-1}\}$

Thus, (3.9) gives $B_B (A_B^*\, A_B)^{r+1} = (A_B^*\, A_B)^r$

Apply (vi) and (vii) of (2.6) repeatedly we obtain,

$B_B A_B^*\, A_B A_B^* = A_B^*$

To show that X_B is unique, we suppose that X_B satisfies (3.7) and (3.8) and that Y_B satisfies,

$Y_B = A_B^* Y_B^* Y_B$ and $A_B^* = A_B^*\, A_B Y_B$

These relations are obtained by substituting (3.6) in (3.4) and (3.5) in (3.3) respectively.

Now, $X_B = X_B X_B^* A_B^*$

$= X_B X_B^* A_B^*\, A_B Y_B$

$= X_B\, A_B Y_B$

$= X_B\, A_B A_B^* Y_B^* Y_B$

$= A_B^* Y_B^* Y_B$

$X_B = Y_B$

The unique solution of (3.3),(3.4),(3.5)and (3.6) is called the generalized inverse of A_B(abbreviated as g.i) and written as $X_B = A_B^\dagger$.

Note 3.8

In the above lemma,

- A_B need not be a square bimatrix and may even be zero.
- Use the notation $\lambda^{\dagger}$ for scalars, where $\lambda^{\dagger}$ means λ^{-1} if $\lambda \neq 0$ and 0 if $\lambda = 0$.

Theorem 3.9

Let $A_B = A_1 \cup A_2$, $B_B = B_1 \cup B_2$ and $C_B = C_1 \cup C_2$ are bimatrices. A necessary and sufficient condition for the equation $A_B X_B B_B = C_B$ to have a solution is,

$$A_B A_B^{\dagger} C_B B_B^{\dagger} B_B = C_B$$

in which case the general solution is,

$$X_B = A_B^{\dagger} C_B B_B^{\dagger} + Y_B - A_B^{\dagger} A_B Y_B B_B B_B^{\dagger},$$

Where Y_B is arbitrary.

Proof

Suppose X_B satisfies $A_B X_B B_B = C_B$

Then $C_B = A_B X_B B_B$

$= A_B A_B^{\dagger} A_B X_B B_B B_B^{\dagger} B_B$

$C_B = A_B A_B^{\dagger} C_B B_B^{\dagger} B_B$

Conversely, if $C_B = A_B A_B^{\dagger} C_B B_B^{\dagger} B_B$, then $A_B^{\dagger} C_B B_B^{\dagger}$ is a particular solution of $A_B X_B B_B = C_B$

For the general solution, we have to solve $A_B X_B B_B = 0$

Now, any expression of the form

$$X_B = Y_B - A_B^{\dagger} Y_B B_B^{\dagger} B_B$$

Satisfies $A_B X_B B_B = 0$.

And conversely if $A_B X_B B_B = 0$ then,

$$X_B = X_B - A_B^{\dagger} A_B X_B B_B B_B^{\dagger}.$$

Theorem 3.10

Let $A_1 X_1 = B_1$ and $A_2 X_2 = B_2$ be two system of equations and can be written as$A_B X_B = B_B$, where $A_B = A_1 \cup A_2$ be a co-efficient bimatrix, $X_B = X \cup Y$ be a unknown bimatrix and $B_B = B_1 \cup B_2$ be a column bimatrix. And let $Aug_B = [A_1 \quad B_1] \cup [A_2 \quad B_2]$ be the augmented bimatrix of the two systems. If the components of augmented bimatrix are equivalent then both the systems has the same solution.

In particular, for the homogeneous system of equations, if the components of co-efficient bimatrix are equivalent then the system must have the unique solution.

Note 3.11

For the non homogeneous system of equations, if the components of augmented bimatrix are not equivalent then both the systems need not have same solution.

Example 3.12

Let the two system of equations be,

$2x_1 + 3x_2 - x_3 = 5\,; \quad x_1 + 2x_2 + x_3 = 8$

$4x_1 + 4x_2 - 3x_3 = 3\,; \quad 2x_1 + 3x_2 + 4x_3 = 20$

$2x_1 - 3x_2 + 2x_3 = 2\,; \quad 4x_1 + x_2 + 2x_3 = 12$

This can be written as

$A_B X_B = B_B$

Where $A_B = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix} \cup \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $B_B = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} \cup \begin{pmatrix} 8 \\ 20 \\ 12 \end{pmatrix}$

The solution is $x_1=1$, $x_2 = 2$, $x_3=3$

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